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ABSTRACT

Higher-order numerical methods derived from polynomial spline interpolation or Hermitian differencing are applied to a separating laminar boundary layer, i.e., the Howarth problem, and the turbulent flat plate boundary layer flow. Preliminary results are presented herein; as in earlier non-separating laminar flow studies, it is found that accuracy equal to that of conventional second-order accurate finite-difference methods is achieved with many fewer mesh points and with reduced computer storage and time requirements.

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Introduction

In a series of recent reports [1-4] polynomial spline and Hermite interpolation procedures have been developed and applied to laminar boundary layer and incompressible Navier-Stokes problems. The polynomial interpolation development is quite general so that a variety of second, fourth and sixth-order spatially accurate representations for both convection and diffusion terms are possible; e.g., a quadratic polynomial leads to the familiar central finite-difference discretization formulas [2]. A detailed discussion of the various formulations with comparisons, interrelationships and examples is given in Refs. (1-4). All of the governing systems are block tridiagonal; the unknowns at each mesh point η_j are at most the function V_j and the spline approximations m_j, M_j of the derivatives $(V_\eta)_j$, $(V_{\eta\eta})_j$, respectively. For the second-order procedures, the governing system can be reduced to include only V_j; for the fourth-order procedures, a reduction requiring only V_j and M_j is possible, and for the sixth-order development the complete 3x3 block tridiagonal system is required.

In the present note, the applicability of the polynomial spline methods for a laminar separating flow, i.e., the linearly decelerating Howarth problem, and the turbulent boundary layer over a flat plate is investigated. In the latter case, the Cebeci-Smith two-layer eddy viscosity model is used for turbulent closure. For both the Howarth and turbulent flat plate flows the solutions have been studied extensively by many investigators so that meaningful comparisons with the present calculations are possible. These two problems

provide somewhat different and in certain ways more severe tests of the polynomial interpolation methods than did previous boundary layer calculations, see Refs. (2-4). From these earlier laminar flow studies, it was found that the fourth-order spline methods generally required one-quarter the number of grid points, in each coordinate direction, as did second-order central finite-difference methods in order to achieve equal accuracy [3,4]. This can lead to reductions of from 50% to 75% in computer time and storage [3,4]. The present calculations will determine the gains, if any, when points of zero shear are approached and when a thick turbulent boundary layer with a thin high shear viscous sublayer is to be approximated.

Four of the polynomial interpolation procedures [1-4] will be considered here: (1) Second-order central differences; (2) a mixed second and fourth-order method termed Spline 2, (3) a fourth-order system termed Spline 4, and (4) a sixth-order Hermite 6 development. The block tridiagonal governing equations are presented in the following sections. For details of the derivations, see Refs. (2-4).

Governing Equations

After appropriate normalization and transformation, the governing equations for both the Howarth flow and turbulent flat plate boundary layer become [5]

$$(1+\epsilon)V_{\eta\eta}+(\epsilon_\eta+f+2\xi f_\xi)V_\eta+\beta(1-V^2)=2\xi VV_\xi, \tag{1}$$

$$f_\eta=V, \tag{2}$$

where $V=V(\xi,\eta)=u/u_e$; u is the streamwise velocity and e denotes po-

tential flow conditions at the boundary layer edge

$$f = f(\xi,\eta),\ u_e = u_e(\xi),\ \beta = \frac{2\xi (u_e)_x}{u_e^2}$$

$$\xi = \xi(x) = \int_0^x u_e(x)\,dx,\ \eta = \eta(x,y) = \frac{y u_e}{(2\nu\xi)^{1/2}},$$

x,y denote the axial and normal coordinate directions; ν is the kinematic viscosity; ϵ is the eddy viscosity for the turbulent boundary layer. The two-layer Cebeci-Smith [6] model is used for ϵ, see Ref. 7. For the Howarth flow: $\epsilon=o$, $u_e = 1-x = (1-2\xi)^{1/2}$, $\beta=2\xi/(1-2\xi)$ For Turbulent flow: u_e=constant, $\beta=o$.

Polynomial Spline Interpolation

Consider a mesh with nodal points at $\eta=\eta_j$. Define the mesh width $h_j=\eta_j-\eta_{j-1}$ and $\sigma_j=h_{j+1}/h_j$. At the mesh points, $V(\xi,\eta)=V(\xi,\eta_j)=V_j$. Define an n^{th} order polynomial $S(\xi,\eta)$ on $[j-1,j]$ such that $S(\xi,\eta_j)=V_j$, and m_j, M_j denote the polynomial approximations of the functional derivatives $(V_\eta)_j$, $(V_{\eta\eta})_j$, respectively. A similar procedure is applied on $[j,j+1]$ and continuity of $S(\xi,\eta)$ and its derivatives is prescribed at η_j. This leads to a coupled system of equations for V_j, m_j, M_j at each mesh point. The system is closed with the differential equation (1) for V_j; functional derivatives are replaced by their polynomial approximations. A similar procedure is used for the function $f(\xi,\eta)$. Finally, in the marching or ξ direction, simple backward differencing is used for V_ξ; i.e., $(V_j^{n+1}-V_j^{n})/\Delta\xi=(V_\xi)_j$, where $\xi=n\Delta\xi$; n=0,1,2,.

For the polynomials considered here, the governing block tri-

diagonal systems for m_j, M_j, with $\sigma_j=\sigma=1$, are as follows. The derivations and results for $\sigma_j\neq 1$ are found in Refs. (1-4).

$$A_j L_{j+1} + B_j L_j + C_j L_{j-1} = D_j, \quad \text{where } L_j = [m, M]_j^T , \tag{3a}$$

$$\text{Central Differencing:} \quad A_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3b}$$

$$\text{Spline 2,4*:} \quad 6A_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ 3B_j = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \ 6C_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3c}$$

$$\text{Hermite 6:} \quad 120A_j = \begin{bmatrix} 45/h & -5 \\ 28 & -4h \end{bmatrix}, \ 1[illegible]_j = \begin{bmatrix} 0 & 5 \\ 8 & 0 \end{bmatrix}, \ 120C_j = \begin{bmatrix} -45/h & -5 \\ 28 & 4h \end{bmatrix} \tag{3d}$$

$$2h^2 D_j = \begin{bmatrix} (V_{j+1}-V_{j-1})h \\ 2(V_{j+1}-2V_j+V_{j-1}) \end{bmatrix} \text{ in all cases.} \tag{3e}$$

Equation (1) becomes

$$(1+\epsilon_j)^{n+1} M_j^{n+1} + (\epsilon_n + f + 2\xi f_\xi)_j^{n+1} m_j^{n+1} + \beta^{n+1}(1-V_j^2)^{n+1} = 2\xi_j V_j^{n+1} \frac{(V_j^{n+1}-V_j^n)}{\Delta\xi} \tag{4}$$

Non-linear terms $(V_j^2)^{n+1}$ are treated by quasi-linearization; i.e., $(V_j^2)^{n+1} = 2V_j^{n+1} V_j^n - (V_j^2)^n$.

The block-tridiagonal system (3a) and (4) are solved by a now standard inversion algorithm [1]. For Spline 2 or 4 the blocks can be reduced to 2x2, while for Hermite 6 they are 3x3. The implicit formulation (4) is unconditionally stable.

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*For Spline 4, $(V_{\eta\eta})_j$ is approximated by $M_j + (M_{j+1}+M_{j-1}-2M_j)/12$, see Refs. (2,3).

Results.

The results for both the Howarth and turbulent flow examples are quite encouraging. Significantly fewer grid points are required to achieve accuracy comparable with that of finite-difference solutions. Even though operational counts are increased due to the larger matrix systems, reductions in computer storage and time are only slightly less than that found in earlier laminar studies [1-4].

Howarth Problem

Typical results are shown on Table 1 and Fig. 1. N denotes the number of mesh points and $h_2=\eta_2$ is the mesh width at the surface. Even with the very coarse grid (h_j=1.0), the Spline 4 result is within 4% of the usually accepted value of x=0.1198 [5]; moreover, the Hermite 6 solution is in error by less than 1%. Also shown on Table 1 are the results [8] of an alternate but somewhat less accurate fourth-order scheme termed here Hermite 4 or compact [8]. This scheme is discussed in Refs. (3, 4, 8). The finite-difference results are in error by more than 20% and require at least four times as many grid points to reduce this error to 4%. It would appear then that separation can be predicted accurately with a minimal number of grid points if polynomial interpolation methods are applied.

N	h_2	σ_j	F.D.	HERMITE 4	SPLINE 4	HERMITE 6
61	0.1	1.0	0.1199	0.1198	0.1198	0.1198
7	1.0	1.0	0.1458	0.1121	0.1159*	0.1193

*If second-order boundary conditions are applied, this value is 0.1191. Since the shear vanishes near separation, and $V_{\eta\eta\eta\eta}(\xi,o) \sim V_\eta(\xi,o)$, these boundary conditions are in fact fourth-order at separation. The spline 4 boundary conditions can be modified to use $V_{\eta\eta\eta\eta}(\xi,o)$ throughout and improve the accuracy of the solution [4,10]. This is important for coarse grids and near separation points.

TABLE I SEPARATION POINT - HOWARTH PROBLEM

Turbulent Flow

Similar trends are observed for the turbulent boundary layer, although considerably larger number of mesh points are required to approximate these thicker high shear regions. Coordinate transformations might facilitate further reductions in the number of points. These were not considered here. Typical results are shown on Table 2 and Figs. 2,3. All indications are that reductions of from one-third (Spline 4) to one-quarter (Hermite 6) the number of mesh points, and 50% in computer time and storage, are possible.

N	h_2	σ_j	Cent. Diff.	Spline 4	Hermite 6
141	0.001	1.063	3.456	3.455	-
61	0.2	1.0	5.479	3.891	3.476
61	0.1	1.0	5.148	-	3.682
21	0.02	$1+4h_j$	4.396	3.472	-
11	0.05	1.5	6.823	3.759	-
11	0.05	$1+4h_j$	-	3.598	-

Wieghardt Data (Ref. 9) $c_f = 3.45 \times 10^{-3}$

$$(c_f)_{laminar} = 0.664 \times 10^{-3}$$

TABLE 2: COEFFICIENT OF FRICTION ($c_f \times 10^3$) at $R_e\ x = 10^6$.

At $R_e\ x = 10^6$ the turbulent flat plate boundary layer is approximately twice as thick as its laminar counterpart and the local shear stress is five times greater. The Spline 4 results with 11 and 21 points are quite good. Hermite 6 solutions are excellent but are given here only for $\sigma_j = 1.0$; the variable grid formulations are given in Ref. 3 and will be applied in future studies. In view of the

present results, the application of higher-order polynomial interpolation for separating and turbulent flows appears to be quite promising. More detailed results and discussion can be found in Ref. 7. In the Appendix attached herein are a number of additional figures from Ref. 7 describing the flow field and solutions for both laminar and turbulent cases. Further analysis of turbulent flows with uniform and variable grid spline 4 and Hermite 6 methods are presented in Ref. 10. Several transformed coordinate systems and flows with surface mass transfer are also considered.

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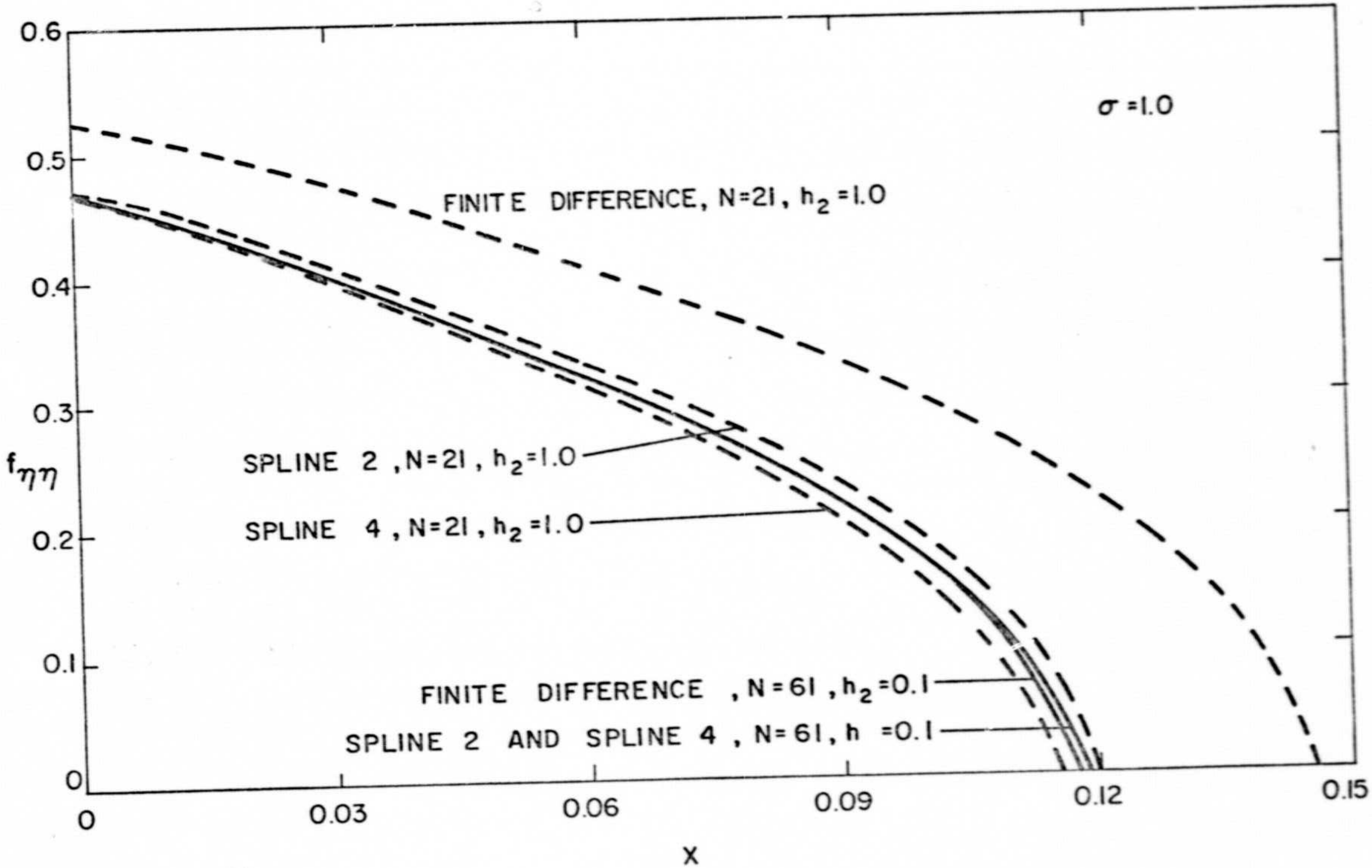


FIG. 1 SHEAR VARIATION ALONG SURFACE (HOWARTH PROBLEM)

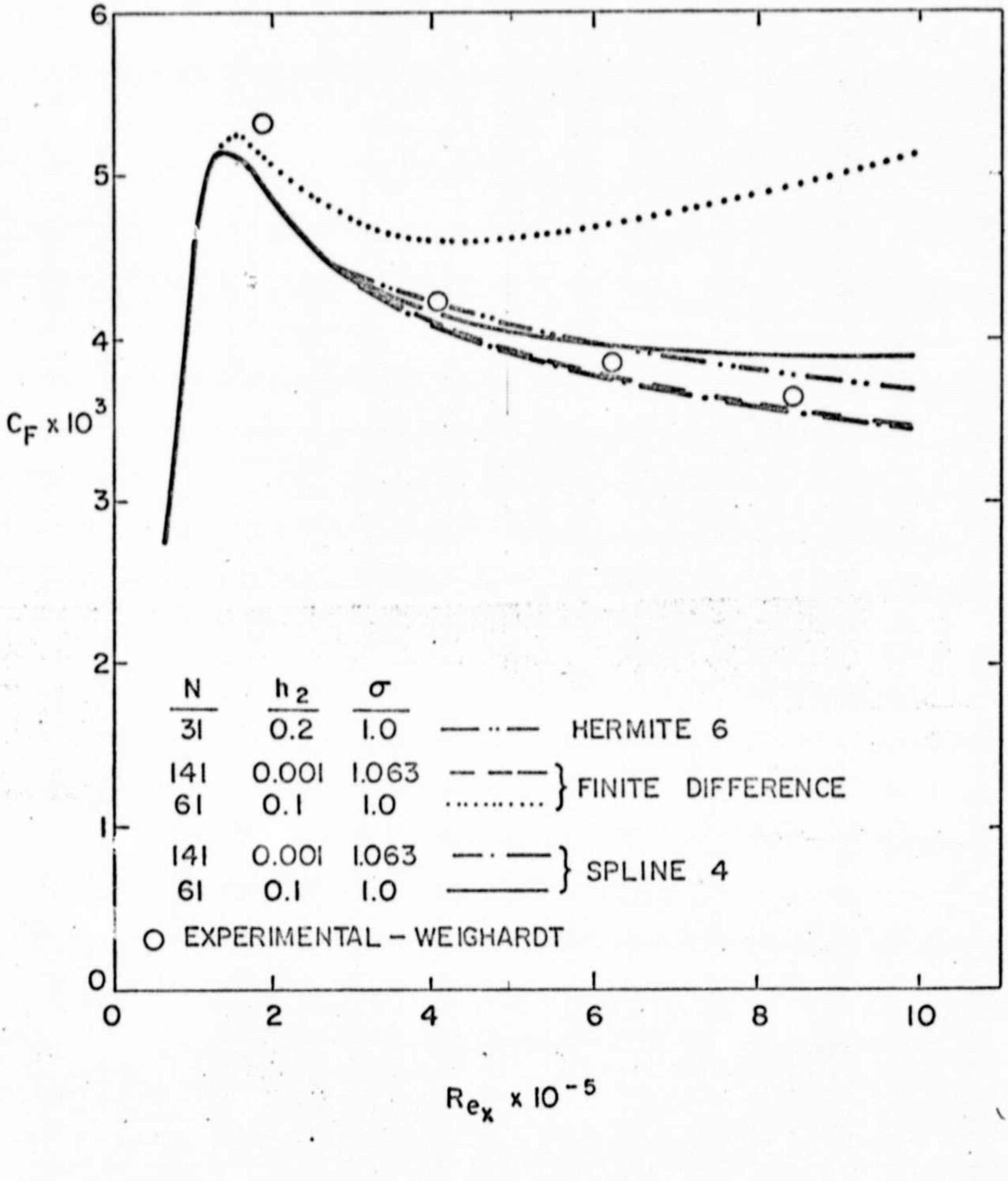


FIG. 2 COEFFICIENT OF FRICTION – TRANSITION REGION

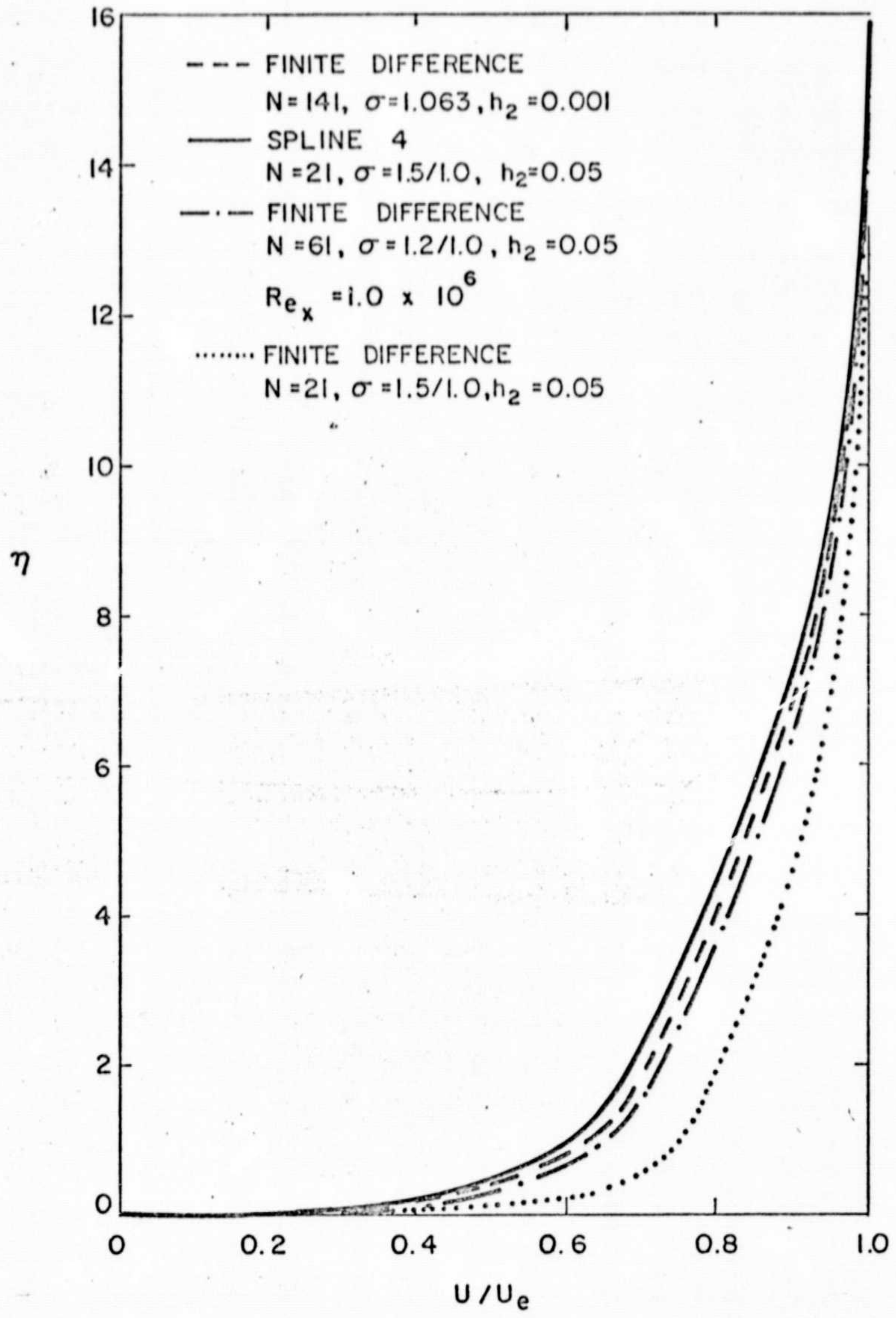
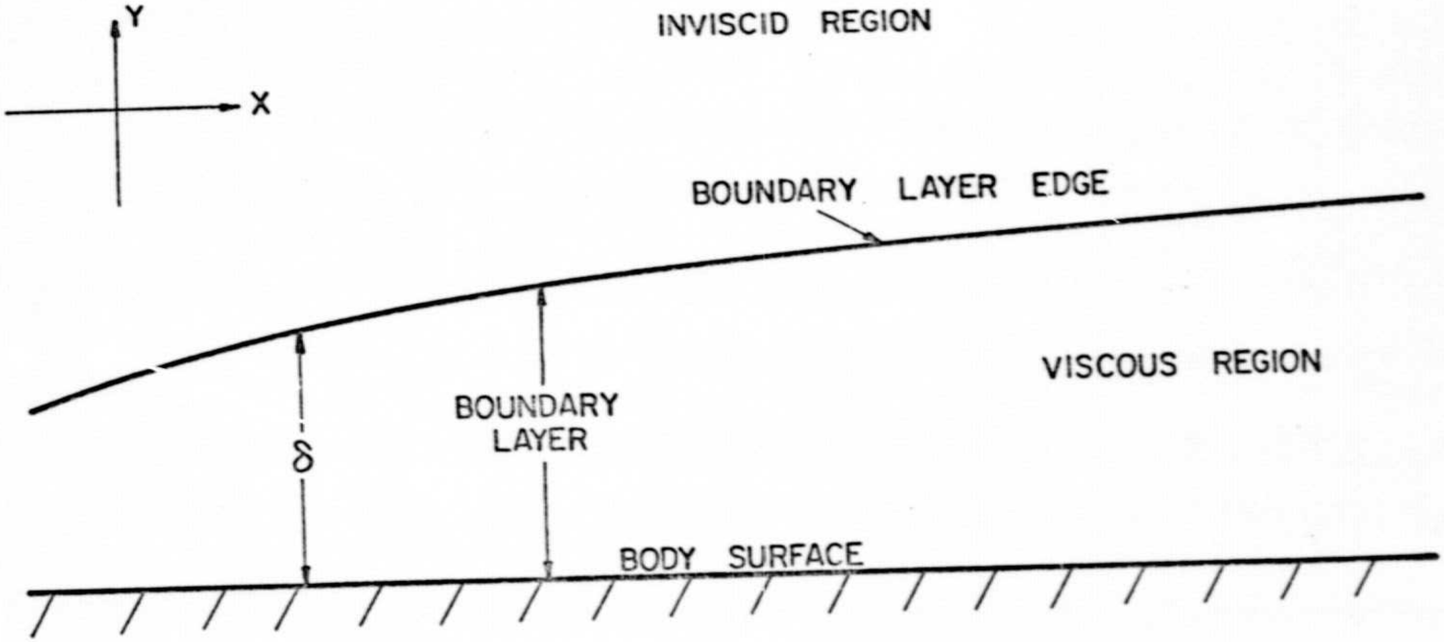
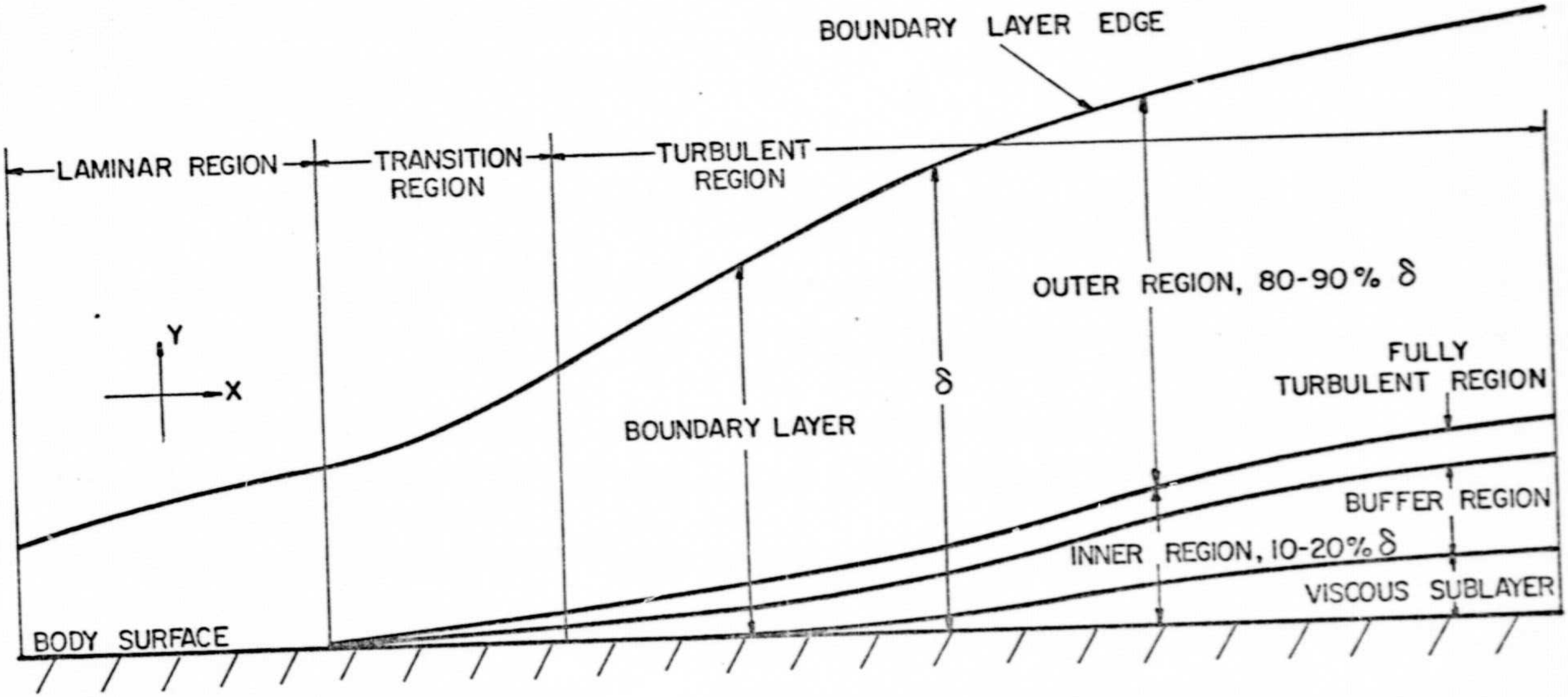


FIG. 3 STREAMWISE VELOCITY VARIATION-TURBULENT BOUNDARY LAYER

APPENDIX: Additional Figures



DESCRIPTION OF LAMINAR BOUNDARY LAYER



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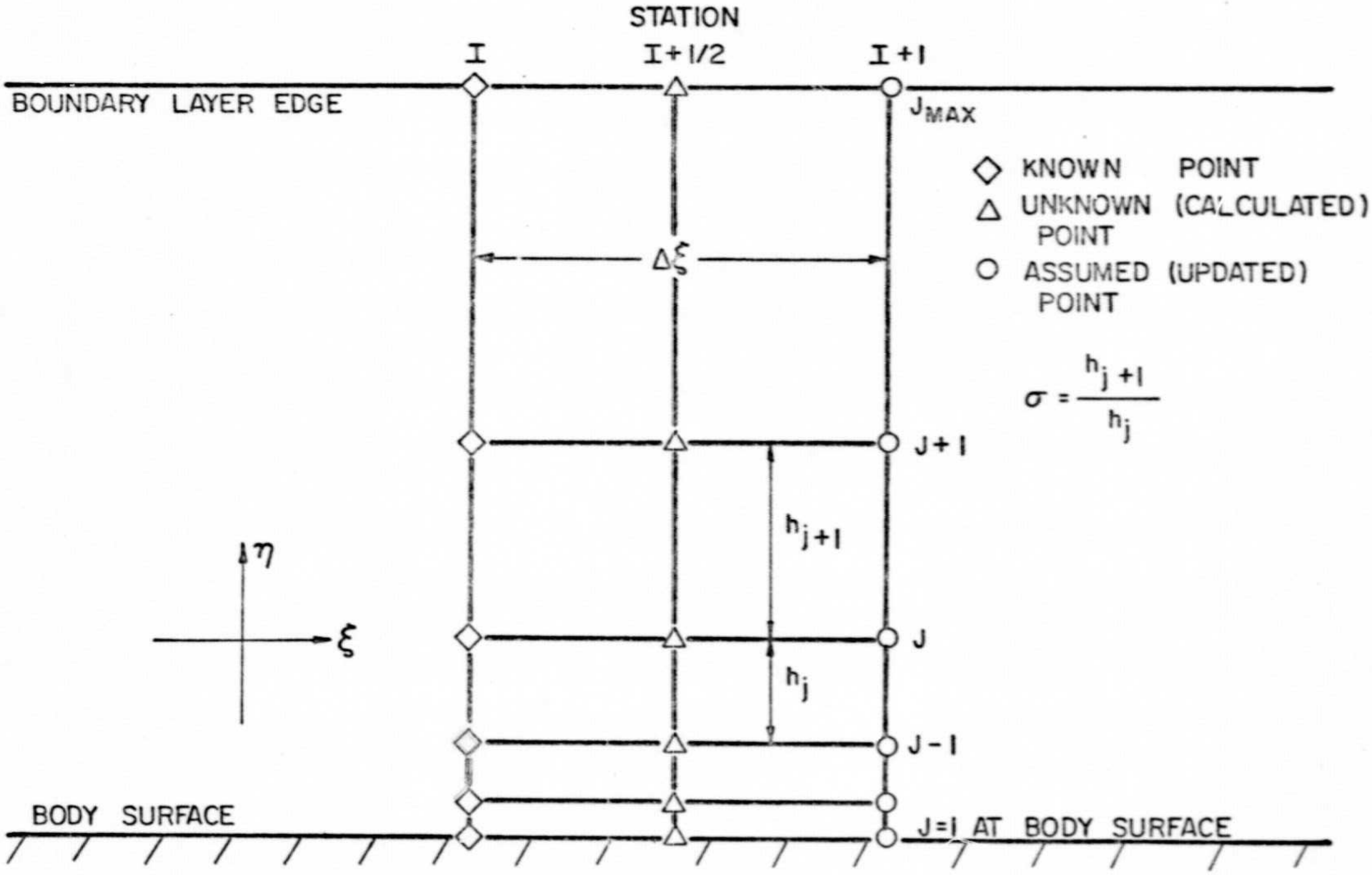
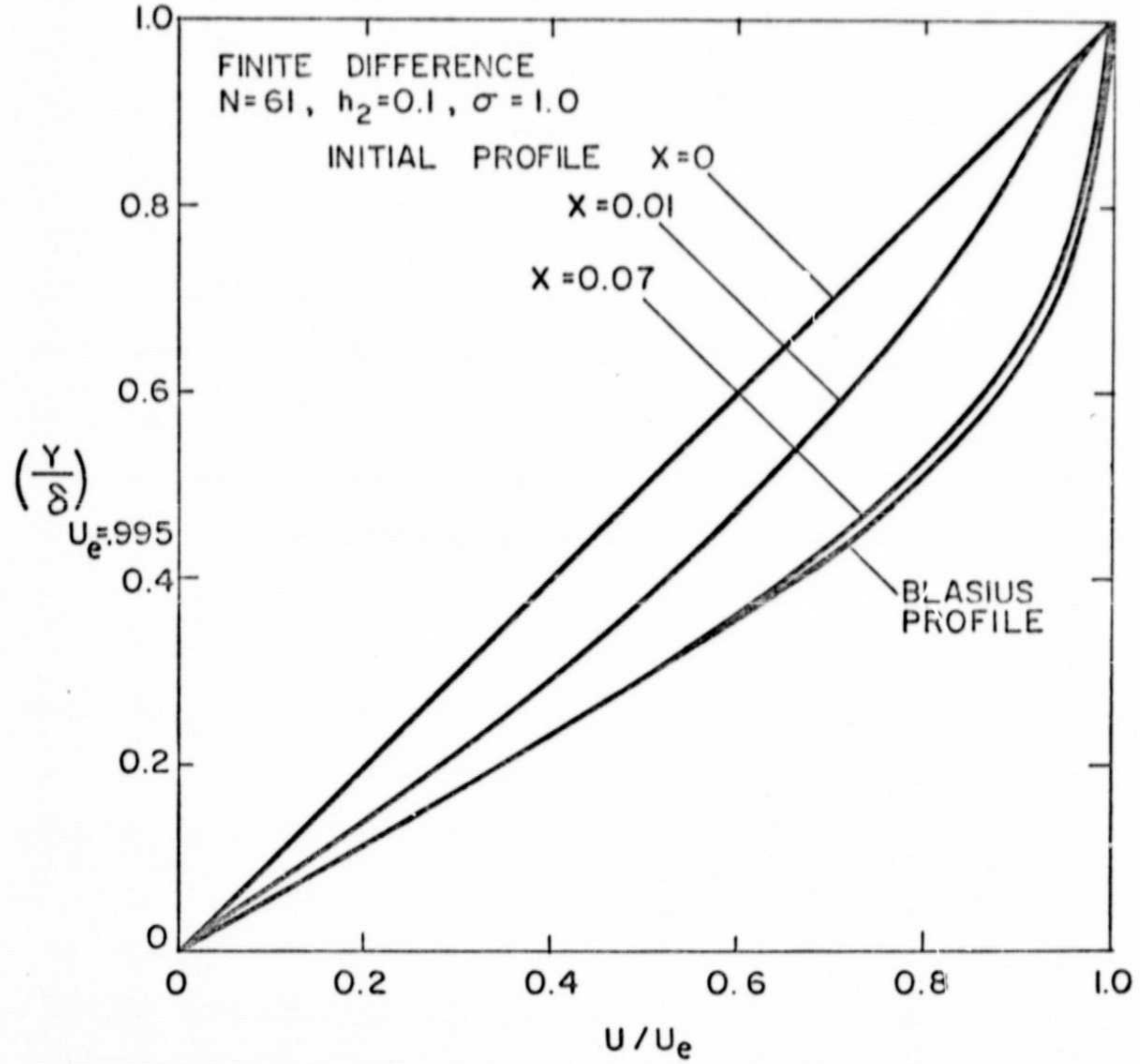
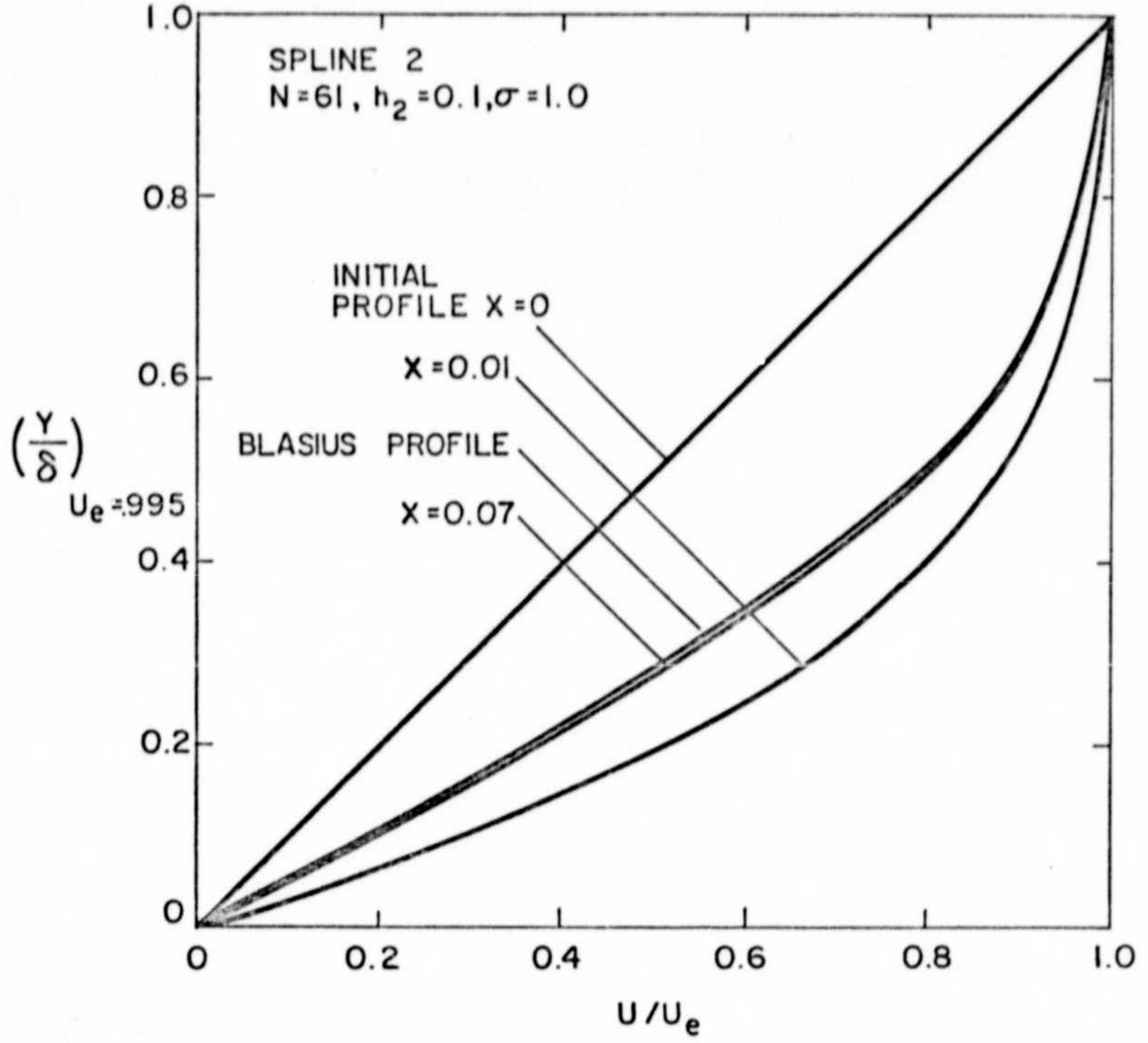


CUBIC SPLINE MESH



STREAMWISE VELOCITY VARIATION IN THE LAMINAR BOUNDARY LAYER WITH ZERO PRESSURE GRADIENT



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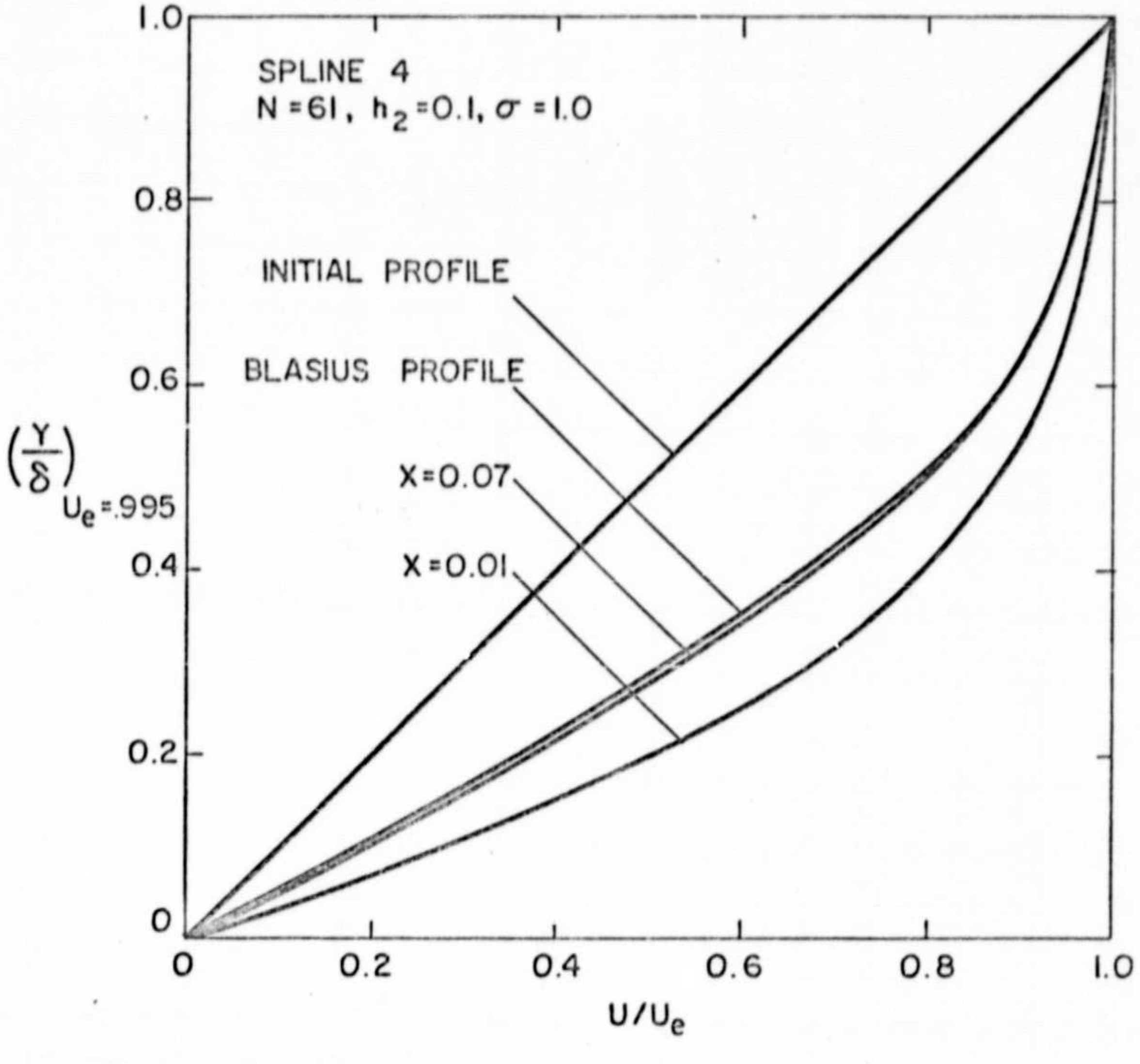

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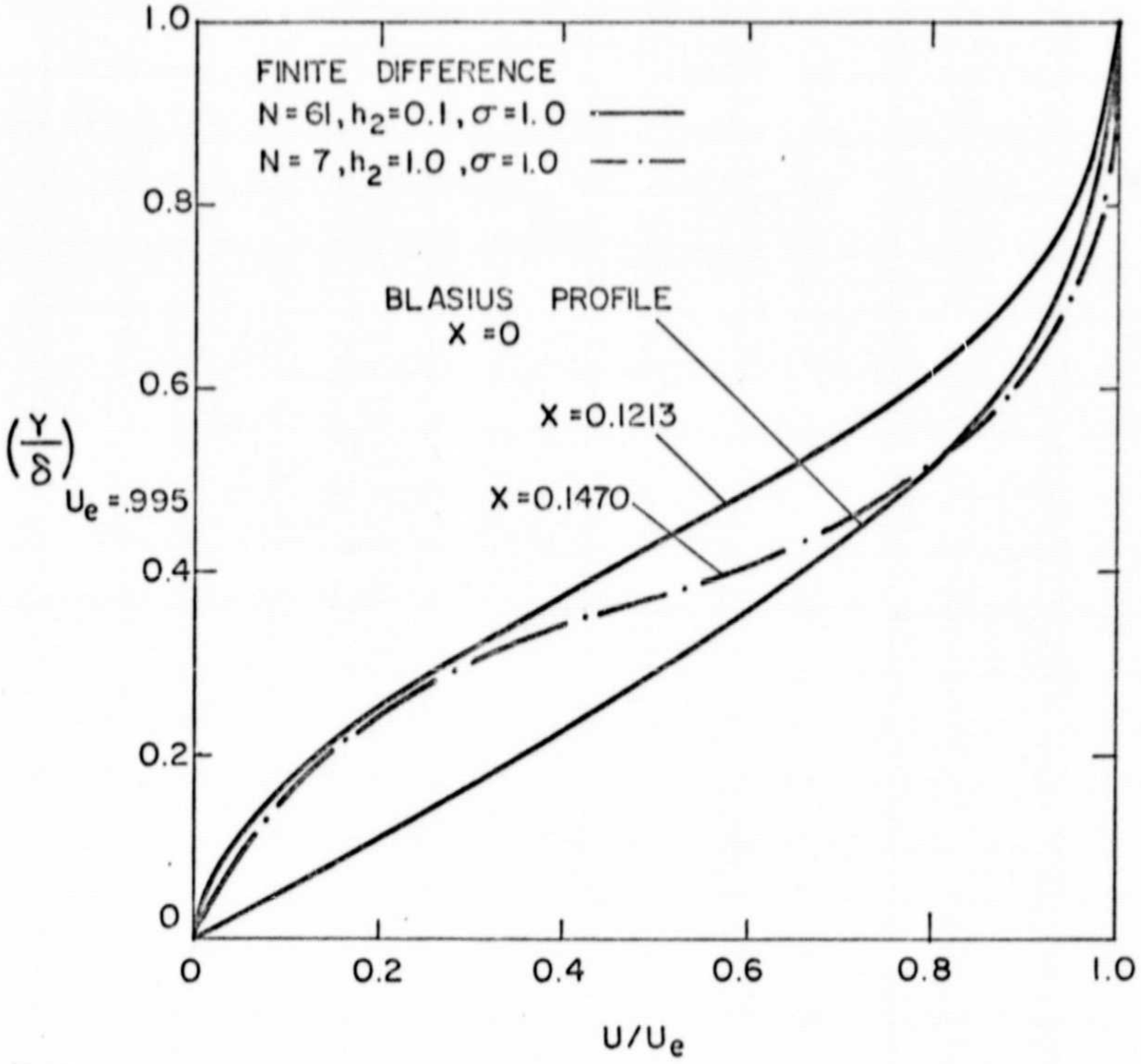

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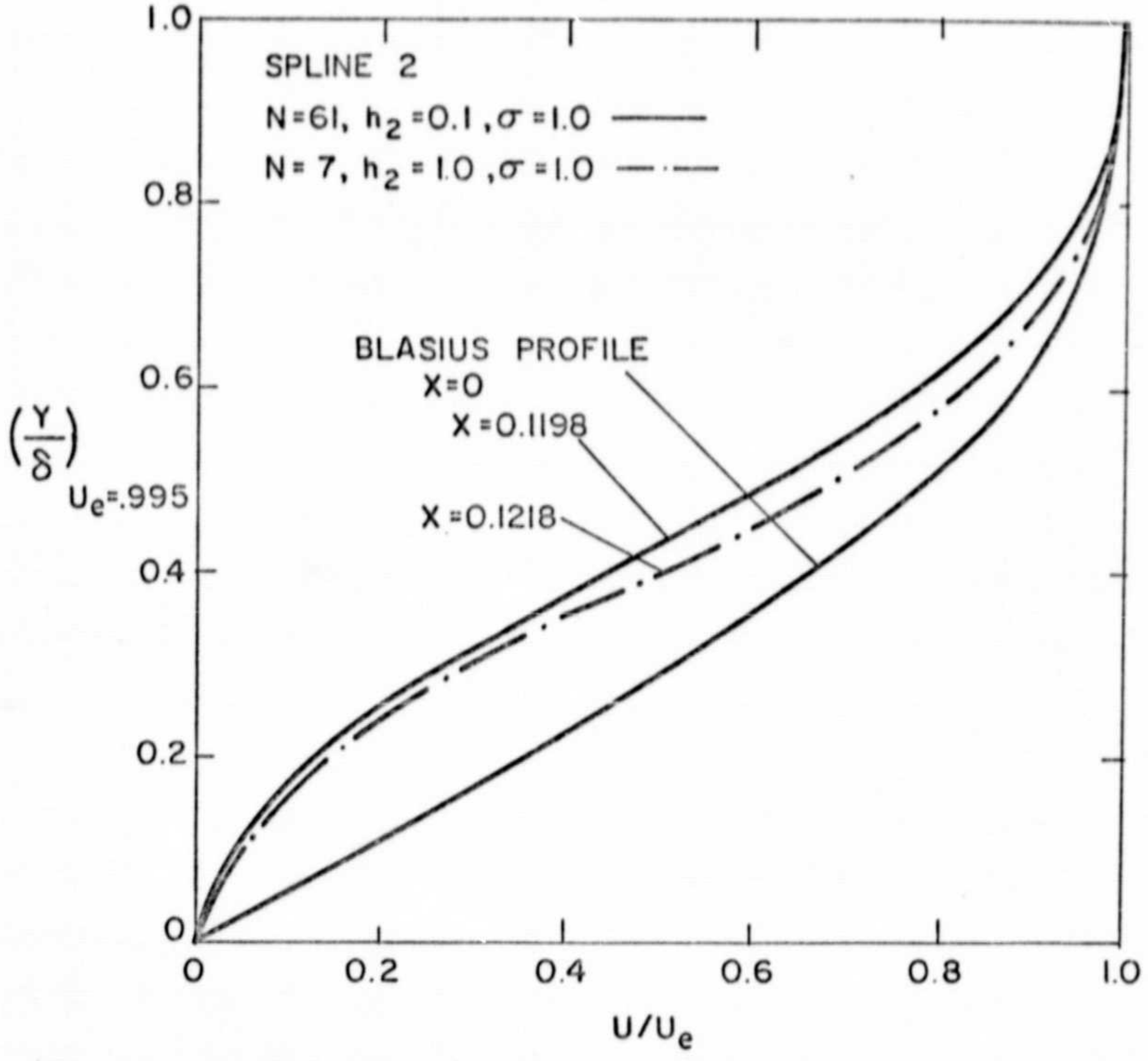

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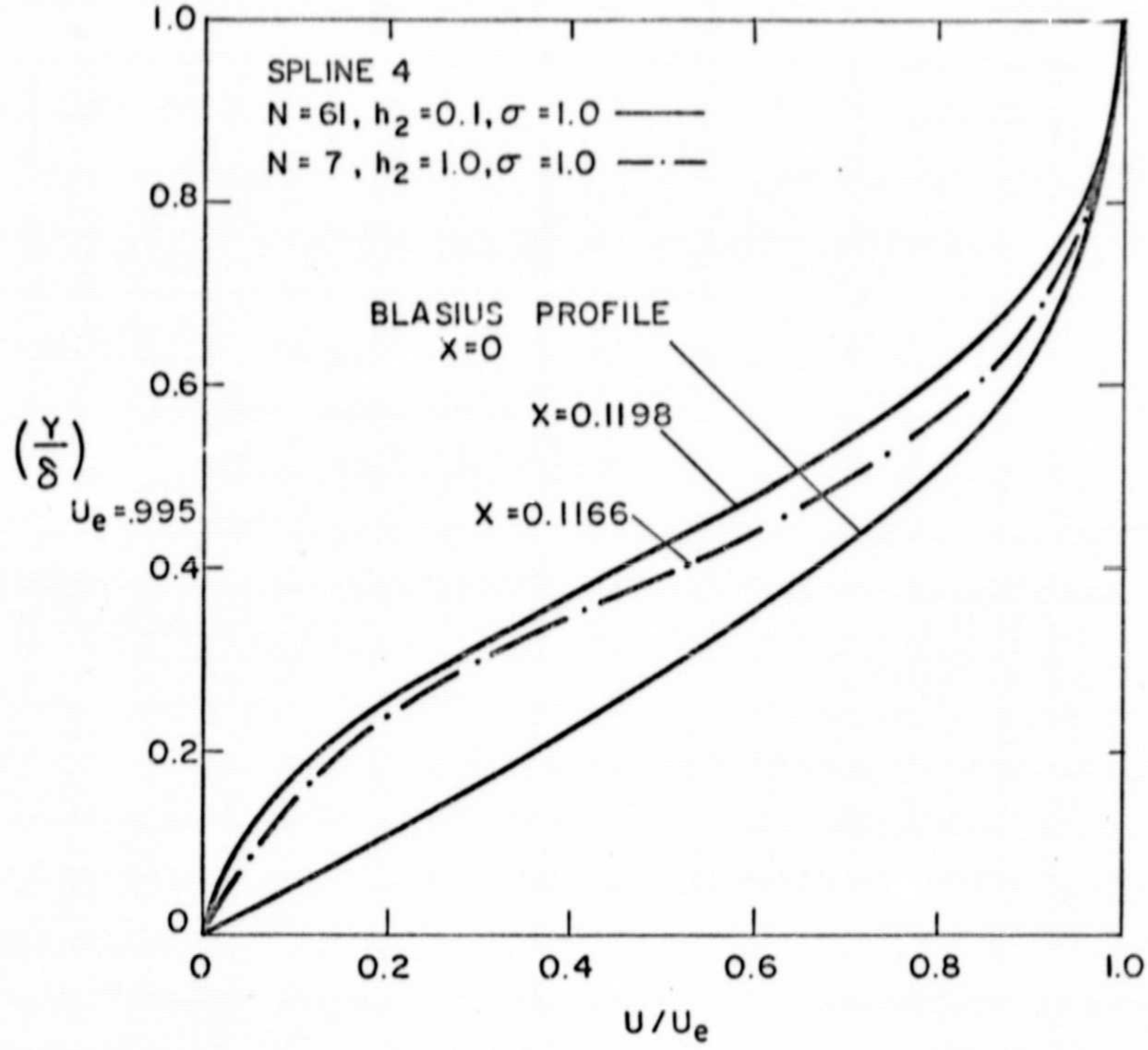

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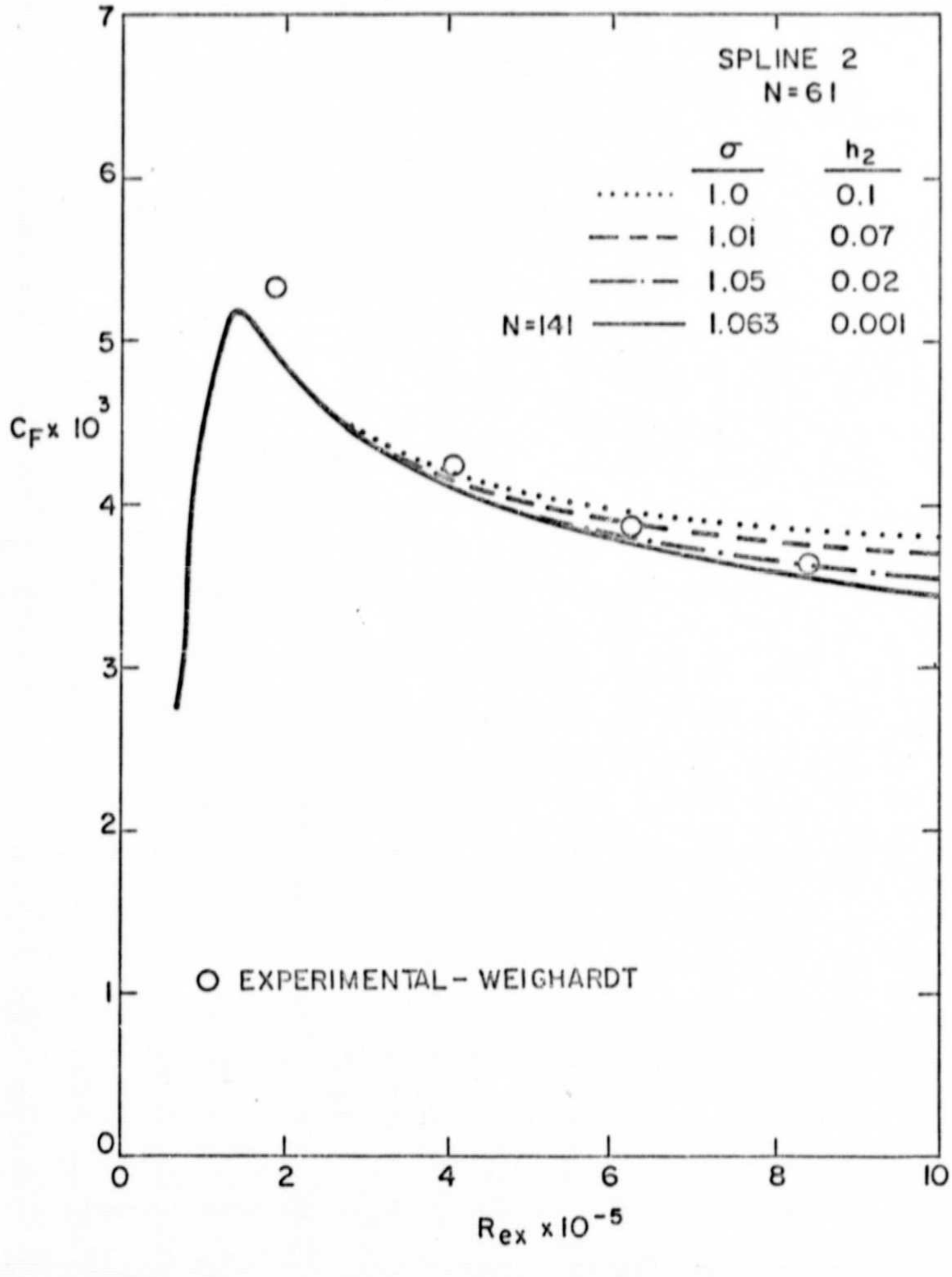
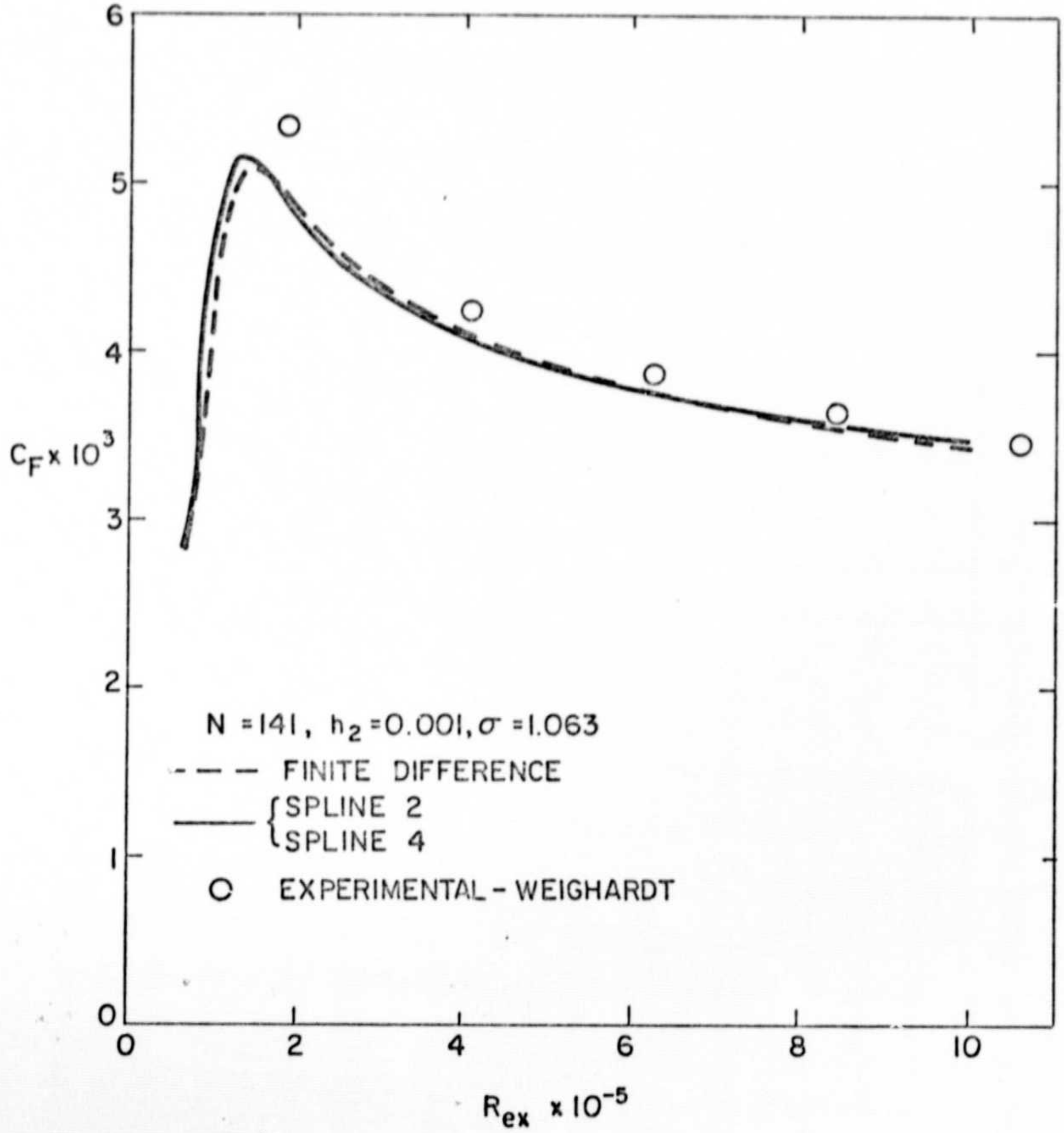


COEFFICIENT OF FRICTION - TRANSITION REGION



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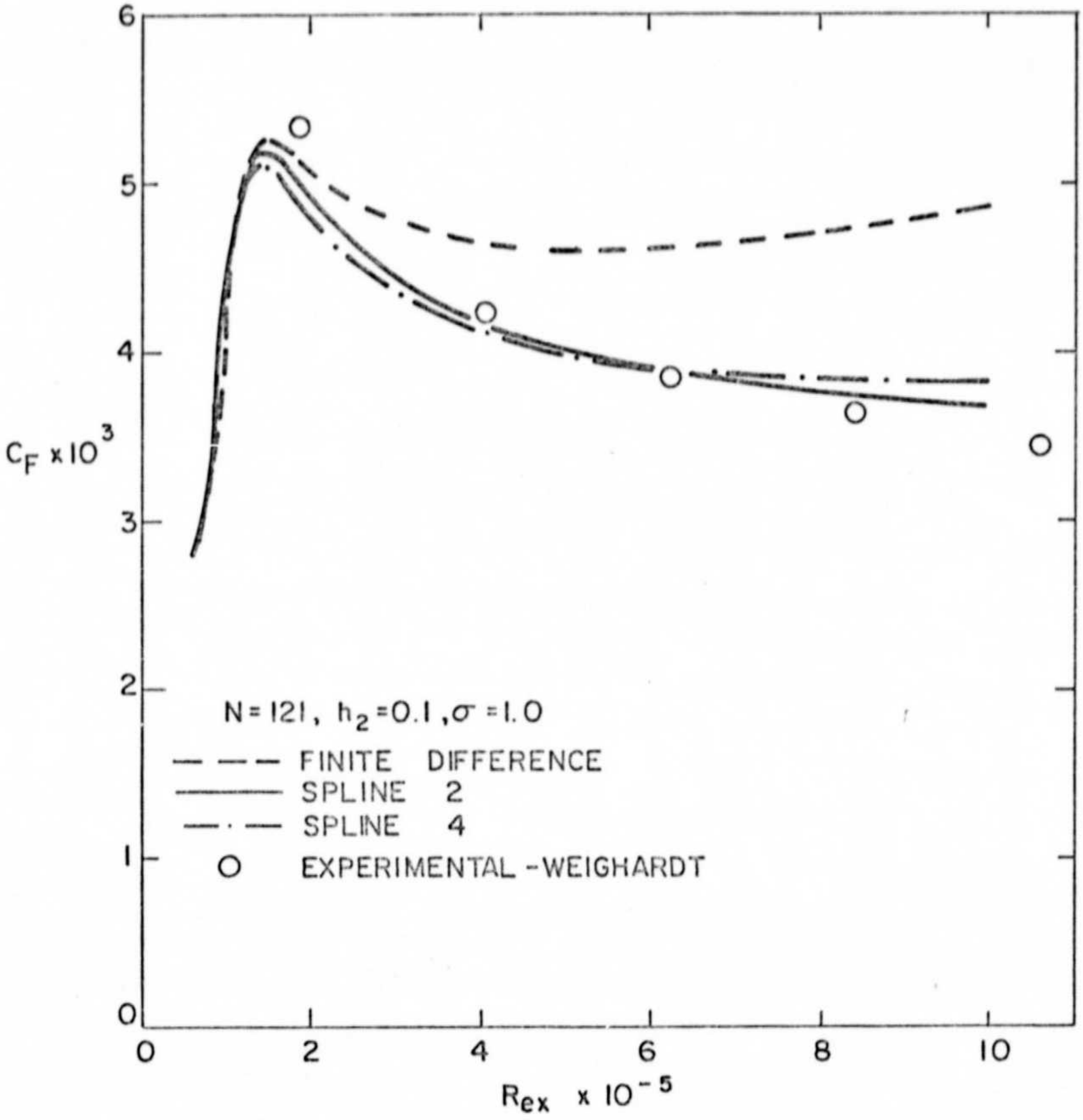

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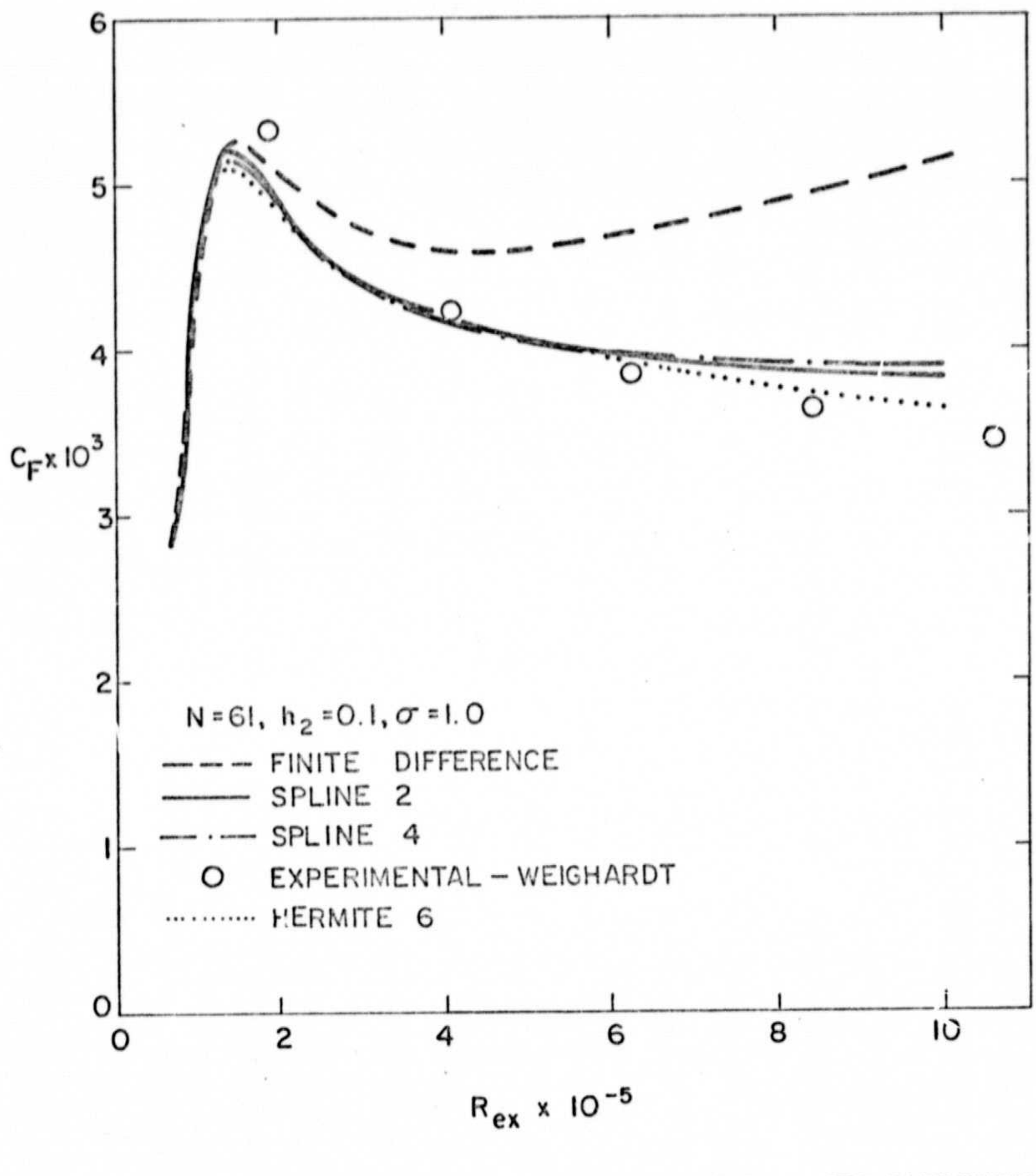

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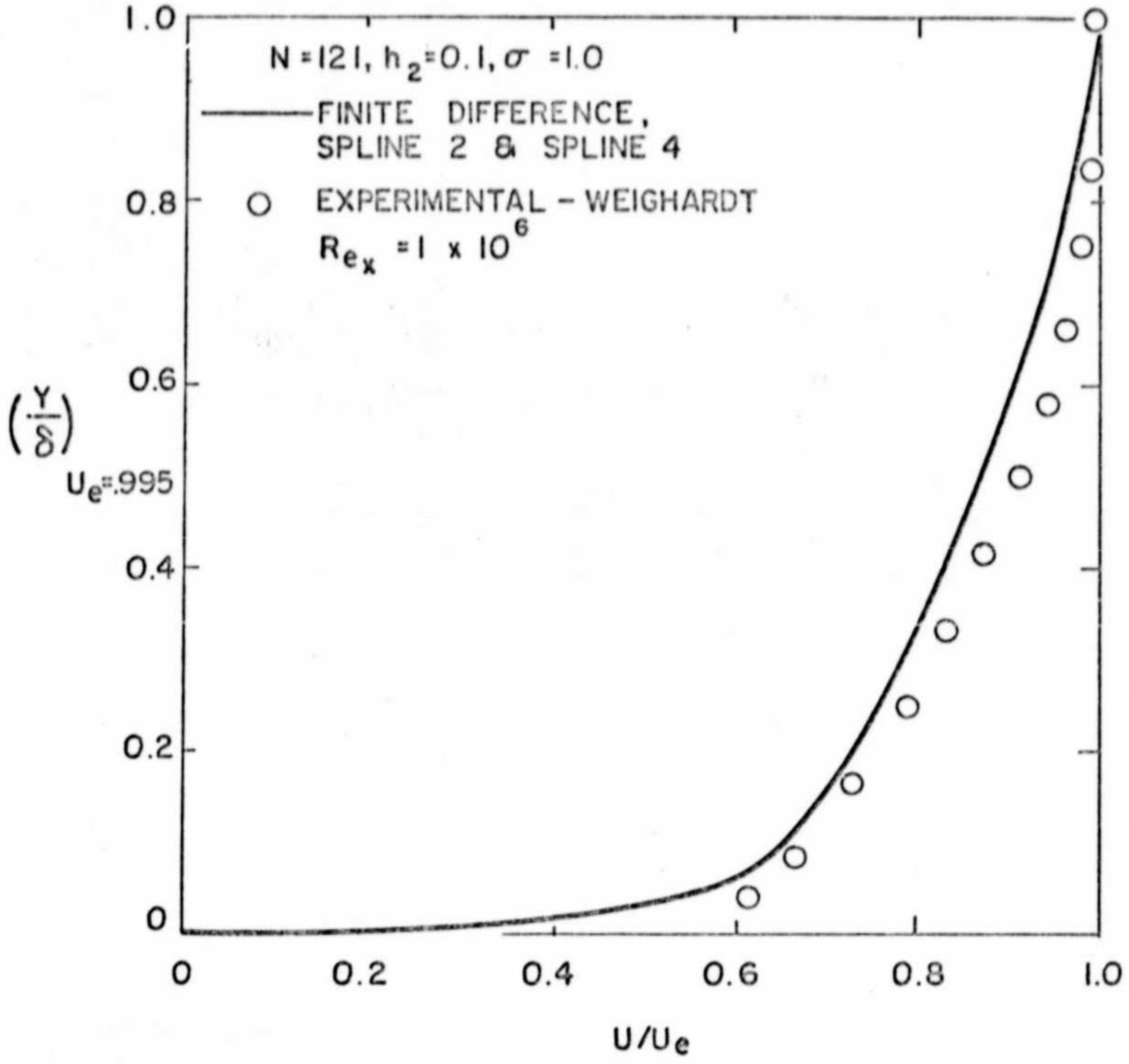

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Hermite interpolation
Howarth flow
Turbulent

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